§ 2.2 - SEPARABLE EQUATIONS

**[B]** An ODE

\[ y' = f(x, y) \]

Is **separable** if we can find \( h \) and \( g \) s.t.

\[ y' = h(x) \cdot g(y) \]

If this is possible, we can write

\[ \frac{y'}{g(y)} = h(x) \quad \text{for} \quad g(y) \neq 0, \]

\[ \Rightarrow \quad \int \frac{y'}{g(y)} \, dx = \int h(x) \, dx \]

Let \( y = y(x) \), \( dy = y'(x) \, dx \)

\[ \Rightarrow \quad \int \frac{dy}{g(y)} = \int h(x) \, dx \]

So as long as we can integrate, we can solve (implicitly).

**[B]** \( \frac{dy}{dx} = \frac{x^2}{1 - y^2} \)

\[ \Rightarrow \quad \int (1 - y^2) \, y' \, dx = \int x^2 \, dx \]

Let \( y = y(x) \), \( dy = y'(x) \, dx \)

\[ \Rightarrow \quad \int 1 - y^2 \, dy = \int x^2 \, dx \]

\[ \Rightarrow \quad y - \frac{y^3}{3} = \frac{x^3}{3} + C \]
Implicit solution
(i.e. not explicit, $y = f(x)$)

It is common to write implicit solutions like this:

$$y - \frac{y^3}{3} - \frac{x^3}{3} = C$$

Because this "u-substitution" always occurs, it is a convenient shortcut to write

$$\frac{dy}{dx} = \frac{x^2}{1-y^2} \Rightarrow (1-y^2) \, dy = x^2 \, dx$$

$$\Rightarrow -x^2 \, dx + (1-y^2) \, dy = 0$$

These all mean the same thing.

$$(4+y^3) \, dy = (4x-x^3) \, dx$$

$$\Rightarrow \int (4+y^3) \, dy = \int (4x-x^3) \, dx$$

$$\Rightarrow 4y + \frac{y^4}{4} = 2x^2 - \frac{x^4}{4} + C$$

$$\Rightarrow 4y + \frac{y^4}{4} = 2x^2 + \frac{x^4}{4} = C$$

Note: Remember, if you divide by something, you should always make a note that what follows assumes the divisor $\neq 0$, and then go back and check what happens if it $= 0$. 